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December, 1957

Scattering of Plane Waves by Locally Homogeneous Dielectric Noise

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Richard A. Silverman

Abstract

We study the scattering of plane waves by a confined region of dielectric noise, using two different approaches. One approach represents the scattering dielectric noise as a finite sample of a homogeneous random process. The other approach represents the dielectric noise as a locally homogeneous random process.

Table of Contents

		Page
1.	Introduction	1
2.	Scattering by a finite volume of homogeneous dielectric noise	1
3.	Scattering by locally homogeneous dielectric noise	8
4.	Conclusion	12
	References	13

1. Introduction

The scattering of acoustic and electromagnetic radiation by random refractive index fluctuations of the scattering medium (e.g., the troposphere, the ocean) is a problem of considerable contemporary interest. For brevity, we shall refer to these refractive index fluctuations as dielectric noise. The purpose of this paper is to compare two approaches to the problem of the scattering of incident plane waves by a confined region of dielectric noise. In the first approach, which is the conventional one, the scattering dielectric noise is represented as a 'section' of a homogeneous (i.e., spatially stationary) random process. In the second approach, the scattering dielectric noise is represented as a locally homogeneous random process. It is thought that the second approach, which is based on a recent generalization of the notion of stationarity[1], may be found useful in the applications.

2. Scattering by a finite volume of homogeneous dielectric noise

Let the refractive index at the point $\vec{r}$ be denoted by the random function (process) $n(\vec{r})$, and let the deviation of $n(\vec{r})$ from its mean be denoted by

$$\Delta n(\vec{r}) = n(\vec{r}) - \langle n(\vec{r}) \rangle \quad ,$$

where the angular brackets indicate the operation of ensemble averaging. By definition, the dielectric noise $\Delta n(\vec{r})$ has mean zero. We assume that $\langle n(\vec{r}) \rangle \sim 1$, and that $\Delta n(\vec{r})$ is second order homogeneous, which means that

$$\langle \Delta n(\vec{r}) \Delta n(\vec{r}') \rangle = \langle (\Delta n)^2 \rangle \, C(\vec{r} - \vec{r}') \quad , \tag{1}$$

where the quantity $\langle (\Delta n)^2 \rangle$ is the average dielectric noise power, and $C(\vec{r} - \vec{r}')$ is the correlation function of $\Delta n(\vec{r})$.

Suppose now that a scalar plane wave $f_o \exp(i\vec{k}_o \cdot \vec{r})$ is incident upon a cubical volume $V = d^3$ of dielectric noise described by the homogeneous process $\Delta n(\vec{r})$. (The wave vector $\vec{k}_o$ has magnitude $k = (2\pi/\lambda)$, where λ is the wavelength, and points in the direction of incidence.) Suppose further that the dielectric noise is weak, i.e., that there is negligible multiple scattering of the incident radiation. Then, to a good approximation, the scattered field observed at a point P in the far field of V is given by the real part of the complex quantity

$$f(P) = -\frac{f_o}{2\pi R} k^2 e^{ikR} \int_V e^{i\vec{K}\cdot\vec{r}} \Delta n(\vec{r})\, d\vec{r} \quad , \tag{2}$$

except for an arbitrary phase [2]. For convenience, we choose the origin of coordinates O at the center of V. The distance from O to P is denoted by R, and, by hypothesis, obeys the far-field or Fraunhofer condition $R\lambda \gg d^2$. The scattering wave vector $\vec{K}$ is defined by

$$\vec{K} = \vec{k}_o - \vec{k}_s \quad , \tag{3}$$

where $\vec{k}_s$ has magnitude $k = (2\pi/\lambda)$, and points in the direction from O to P. (See Figure 1.) In the case of sound waves, the field variable f is taken to be the pressure. With a slight modification, equation (2) can be used to describe the scattering of linearly polarized electromagnetic radiation with electric vector in the direction $\vec{a}$. One simply takes f(P) to be the amplitude of the scattered electric field measured in its direction of polarization, and replaces f_o in the right side of (2) by $f_o \sin\beta$, where β is the angle between $\vec{a}$ and $\vec{k}_s$ [3], [4].

It is always the case in practice that the scattering volume V can be divided into a large number M of subvolumes which make substantially independent contributions to f(P). In particular $M \gtrsim (d/\ell)^3$, where the correlation distance ℓ is some reasonable measure of the distance over which the

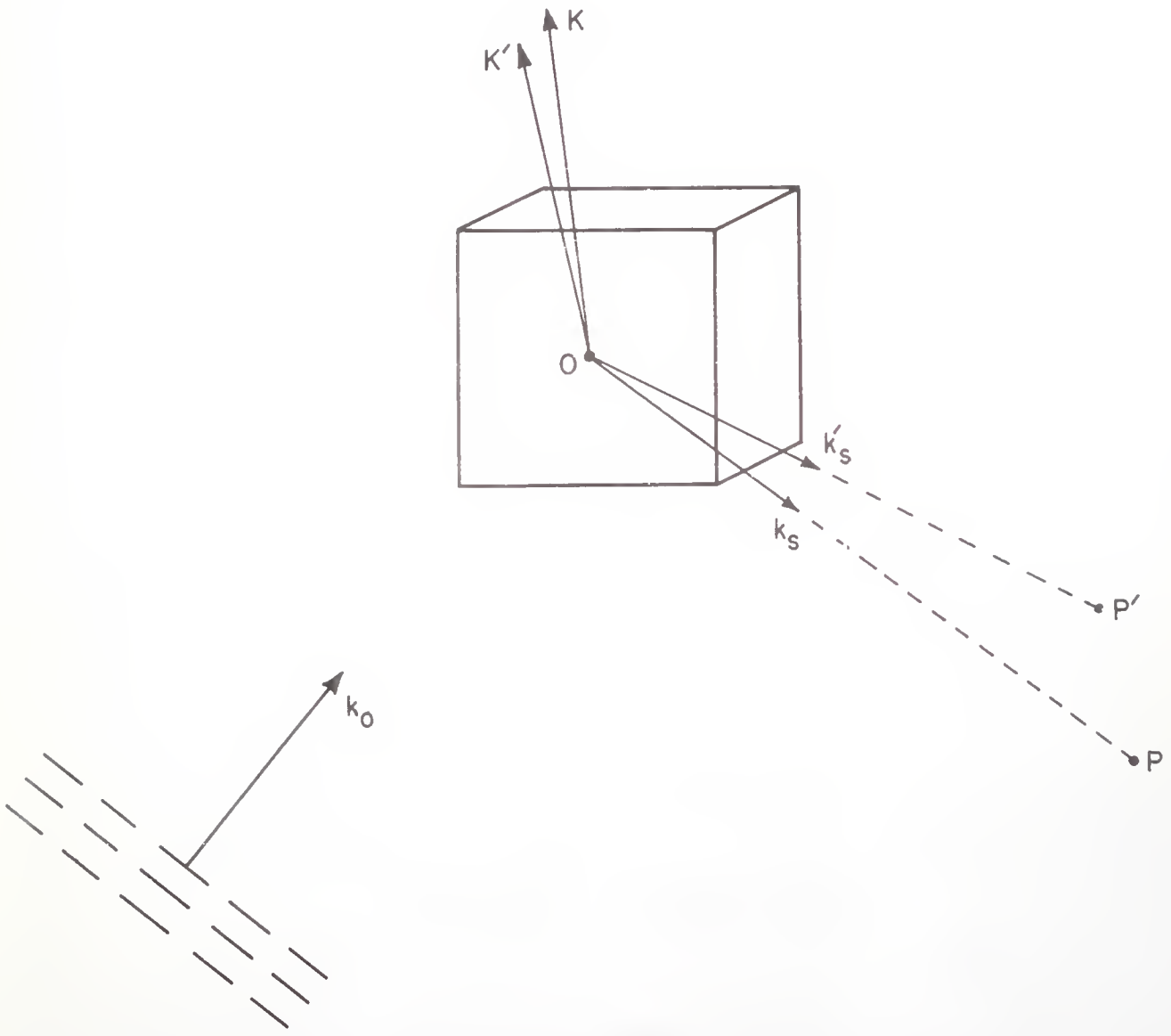

Figure 1. The scattering configuration (schematic)

process $\Delta n(\vec{r})$ is appreciably auto-correlated; we assume that $d \gg \ell$. It follows by a multivariate Central Limit argument that $f(P)$ is an approximately Gaussian random function of P, in the sense that the distribution of the m random variables

$$f(P_1), \ldots, f(P_m)$$

is m-variate, complex Gaussian, to a good approximation, provided that $m \ll M$.* To the extent that $f(P)$ is a Gaussian process, its complete statistical characterization is contained in its covariance $\langle f(P) f^*(P') \rangle$. (The asterisk denotes the complex conjugate.) By (1) and (2), we have

$$(4) \qquad \langle f(P) f^*(P') \rangle = \frac{f_o^2 \langle (\Delta n)^2 \rangle}{4\pi^2 R R'} \, k^4 \, e^{ik\Delta R} \int_V \int_V e^{i(\vec{K}\cdot\vec{r} - \vec{K}'\cdot\vec{r}')} C(\vec{r}-\vec{r}') d\vec{r} d\vec{r}',$$

where R' is the distance from O to P', another point in the far-field of V, and $\Delta R = R - R'$. As indicated, the variables of integration $\vec{r}$ and $\vec{r}'$ separately range over V. The scattering wave vector $\vec{K}$ is defined by (3), and the scattering wave vector $\vec{K}'$ by

$$(3a) \qquad \vec{K}' = \vec{k}_o - \vec{k}_s' \, ,$$

where $\vec{k}_s'$ has magnitude $(2\pi/\lambda)$, and points in the direction from O to P'. (See Figure 1.) The integral in the right side of (4) can be written either as

$$(5) \qquad \int_V \int_V e^{i\vec{K}\cdot(\vec{r}-\vec{r}')} \, e^{i(\vec{K}-\vec{K}')\cdot\vec{r}'} \, C(\vec{r}-\vec{r}') \, d\vec{r} \, d\vec{r}' \, ,$$

or as

$$(5a) \qquad \int_V \int_V e^{i\vec{K}'\cdot(\vec{r}-\vec{r}')} \, e^{i(\vec{K}-\vec{K}')\cdot\vec{r}} \, C(\vec{r}-\vec{r}') \, d\vec{r} \, d\vec{r}' \, .$$

---

*If $\Delta n(\vec{r})$ is known to be a Gaussian process, then the Central Limit argument is not needed, since a linear transform of a Gaussian process is automatically a Gaussian process. However, in some applications, it is suspected that $\Delta n(\vec{r})$, which may be due, for example, to turbulent mixing of refractive index inhomogeneities, is non-Gaussian.

Introducing new variables of integration $\vec{\rho} = \vec{r} - \vec{r}'$, $\vec{\eta} = \vec{r}'$ in the first case, and $\vec{\rho} = \vec{r} - \vec{r}'$, $\vec{\eta} = \vec{r}$ in the second case, we find that the integrals (5) and (5a) become

$$\int_W e^{i\vec{K}\cdot\vec{\rho}}\, e^{i(\vec{K}-\vec{K}')\cdot\vec{\eta}}\, C(\vec{\rho})\, d\vec{\rho}\, d\vec{\eta} \quad , \tag{6}$$

and

$$\int_{W'} e^{i\vec{K}'\cdot\vec{\rho}}\, e^{i(\vec{K}-\vec{K}')\cdot\vec{\eta}}\, C(\vec{\rho})\, d\vec{\rho}\, d\vec{\eta} \quad . \tag{6a}$$

The six-dimensional regions of integration W and W' can be simply described by giving their projections on the (ρ_x, η_x) plane. (Their projections on the (ρ_x, η_x), (ρ_y, η_y), and (ρ_z, η_z) planes have an identical appearance.) Thus, the parallelogram in Figure 2a shows the projection of W on the (ρ_x, η_x) plane, and the parallelogram in Figure 2b shows the corresponding projection of W'.

To approximate the integrals (6) and (6a), we make use of the assumption $d \gg \ell$, already used in connection with the Central Limit argument. Thus, $C(\vec{\rho})$ will be substantially different from zero only in the strips of width ℓ shown in Figure 2. It is therefore a good approximation to write (6) and (6a) as

$$\int_{-\infty}^{\infty} e^{i\vec{K}\cdot\vec{\rho}}\, C(\vec{\rho})\, d\vec{\rho} \int_V e^{i(\vec{K}-\vec{K}')\cdot\vec{\eta}}\, d\vec{\eta} \quad , \tag{7}$$

and

$$\int_{-\infty}^{\infty} e^{i\vec{K}'\cdot\vec{\rho}}\, C(\vec{\rho})\, d\vec{\rho} \int_V e^{i(\vec{K}-\vec{K}')\cdot\vec{\eta}}\, d\vec{\eta} \quad , \tag{7a}$$

where, as indicated, the integration over $\vec{\rho}$ runs from $-\infty$ to ∞ in all its components. Since $C(\vec{\rho})$ is a correlation function, it is the Fourier transform of a non-negative function $\Phi(\vec{k})$, known as the (fluctuation) spectrum of $\Delta n(\vec{r})$, i.e.,

$$C(\vec{\rho}) = \int_{-\infty}^{\infty} e^{-i\vec{k}\cdot\vec{\rho}}\, \Phi(\vec{k})\, d\vec{k} \quad , \tag{8}$$

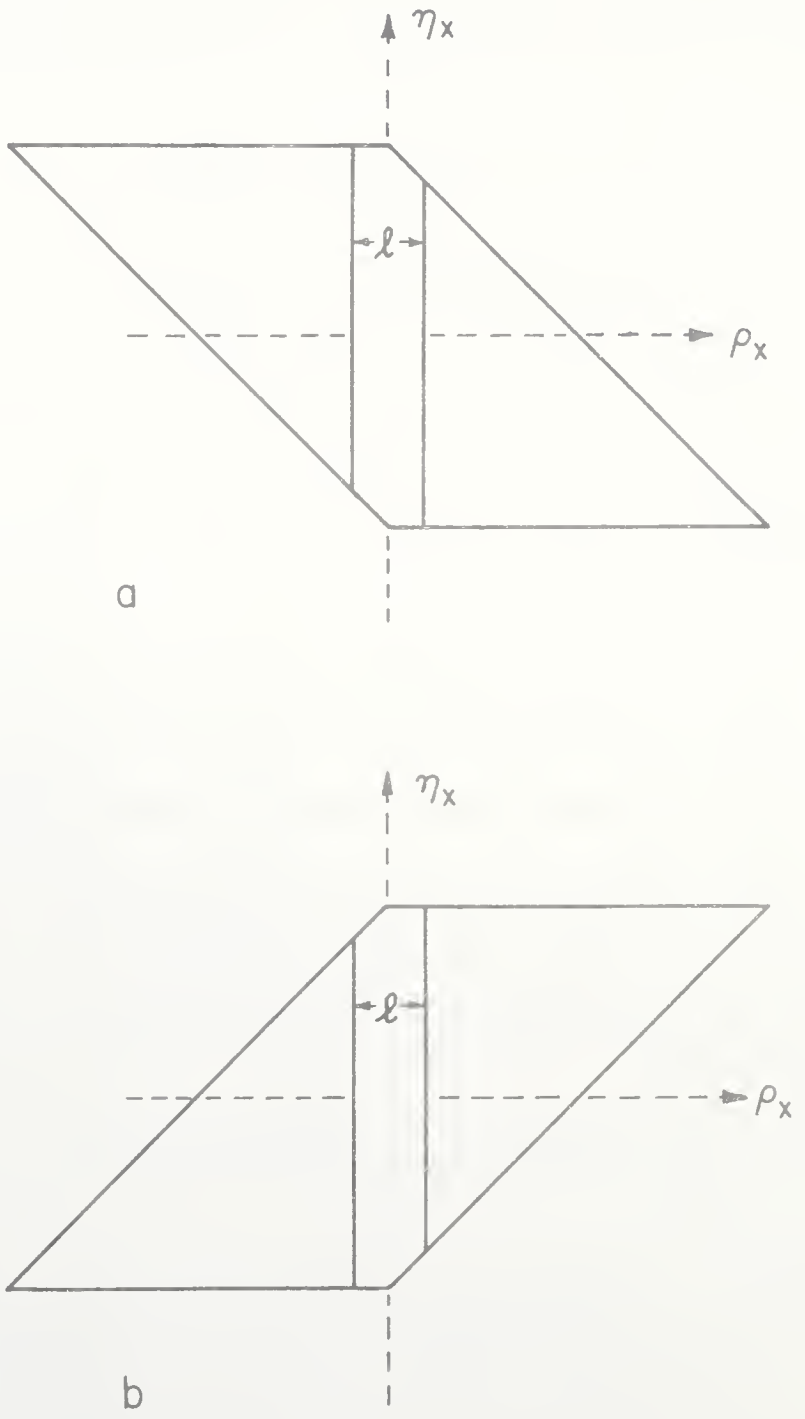

Figure 2. Projections of the regions W and W′

and

$$\Phi(\vec{k}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{i\vec{k}\cdot\vec{\rho}}\, C(\vec{\rho})\, d\vec{\rho} \quad . \tag{8a}$$

Thus, introducing the <u>form factor</u>

$$G(\vec{k}) = \frac{1}{V} \int_V e^{i\vec{k}\cdot\vec{\rho}}\, d\vec{\rho} = \frac{\sin(k_x d/2)}{k_x d/2}\, \frac{\sin(k_y d/2)}{k_y d/2}\, \frac{\sin(k_z d/2)}{k_z d/2} \quad , \tag{9}$$

itself a correlation function, we finally have two approximations for the covariance of f(P), namely

$$\langle f(P) f^*(P') \rangle \sim \frac{2\pi f_o^2 V \langle (\Delta n)^2 \rangle}{RR'} k^4\, e^{ik\Delta R}\, \Phi(\vec{K})\, G(\vec{K}-\vec{K}') \quad , \tag{10}$$

and

$$\langle f(P) f^*(P') \rangle \sim \frac{2\pi f_o^2 V \langle (\Delta n)^2 \rangle}{RR'} k^4\, e^{ik\Delta R}\, \Phi(\vec{K}')\, G(\vec{K}-\vec{K}') \quad . \tag{10a}$$

(The second of these approximations differs from the first only by having $\Phi(\vec{K}')$ in place of $\Phi(\vec{K})$.) For practical purposes, it is enough to require that (10) and (10a) be good approximations only when the magnitude of $\langle f(P) f^*(P') \rangle$ is an appreciable fraction of its value for $P = P'$. It follows from an examination of the form factor $G(\vec{K}-\vec{K}')$ that for this to be the case, $\vec{K}$ and $\vec{K}'$ must both lie within some cube of volume $V' \sim (2\pi)^3/V$ in k-space. Thus, the approximations (10) and (10a) are consistent provided that

$$\Phi(\vec{K}) \sim \Phi(\vec{K}') \quad , \quad \text{when} \quad |\vec{K}-\vec{K}'| \lesssim \frac{2\pi}{d} \quad . \tag{11}$$

For any reasonable functional form of $\Phi(\vec{k})$, (11) is valid if $Kd \gg 2\pi$, a condition always met in practice. In fact, the weaker condition $K\ell \gg 2\pi$ is usually met in practice, and is the justification for the application of inertial range turbulence theories to the problem of scattering by turbulence-induced dielectric noise.

The similarity of the approximations (10) and (10a) suggests that they be combined in the symmetric form

$$(12) \qquad \langle f(P) f^*(P') \rangle \sim \frac{2\pi f_o^2 V \langle (\Delta n)^2 \rangle}{RR'} k^4 e^{ik\Delta R} \Phi\left(\frac{\vec{K}+\vec{K}'}{2}\right) G(\vec{K}-\vec{K}') \quad .$$

The significance of this form of the approximation will become apparent in the next section. It is easy to derive from (12) the usual results pertaining to the space diversity properties of the scattered radiation in the far-field approximation. In fact, (12) shows that the pattern of space correlations of the scattered radiation may be regarded as the Fraunhofer diffraction pattern of the scattering volume V, and has nothing to do with the detailed structure of the dielectric noise. When $P = P'$, (12) becomes

$$\langle |f(P)|^2 \rangle \sim \frac{2\pi f_o^2 V \langle (\Delta n)^2 \rangle}{R^2} k^4 \Phi(\vec{K}) \quad ,$$

which shows that $\Phi(\vec{K})$ is proportional to the average scattered power received at a fixed point P.

Another derivation of (10), (10a), and (12) can be given by using the harmonic analysis of the process $\Delta n(\vec{r})$. The idea is to take the approximation

$$(2) \qquad f(P) = -\frac{f_o V}{2\pi R} k^2 e^{ikR} \frac{1}{V} \int_V e^{i\vec{K}\cdot\vec{r}} \Delta n(\vec{r}) \, d\vec{r} \quad ,$$

expand the process $\Delta n(\vec{r})$ as a stochastic Fourier-Stieltjes integral, and perform the integration with respect to $\vec{r}$ <u>before</u> taking the covariance of f(P). Thus, we begin by writing

$$(13) \qquad \Delta n(\vec{r}) = \int_{-\infty}^{\infty} e^{-i\vec{k}\cdot\vec{r}} \, dN(\vec{k}) \quad ,$$

where the Fourier-Stieltjes integral (13) is meant in the mean-square sense [5], and the integration extends over all k-space, as indicated by the infinite limits.

It follows from the homogeneity of the process $\Delta n(\vec{r})$ that the increments of the process $N(\vec{k})$ obey the relation

$$\langle dN(\vec{k})dN^*(\vec{k}')\rangle = \langle(\Delta n)^2\rangle \, \underline{\Phi}(\vec{k}) \, \delta(\vec{k}-\vec{k}') \, d\vec{k} \, d\vec{k}' \quad , \tag{14}$$

where $\underline{\Phi}(\vec{k})$ is the fluctuation spectrum defined by (8) and (8a), and $\delta(\vec{k}-\vec{k}')$ is the Dirac delta function. Using (13) and (14), we see that the integral in the right side of (2) is

$$\frac{1}{V}\int_V e^{i\vec{K}\cdot\vec{r}}\Delta n(\vec{r})d\vec{r} = \int_{-\infty}^{\infty} dN(\vec{k}) \, \frac{1}{V}\int_V e^{i(\vec{K}-\vec{k})\cdot\vec{r}}d\vec{r} = \int_{-\infty}^{\infty} G(\vec{K}-\vec{k})dN(\vec{k}), \tag{15}$$

where $G(\vec{K}-\vec{k})$ is the form factor defined by (9). Equation (15) shows that the scattered signal is a weighted linear combination, actually a convolution, of the (orthogonal) increments of the process $N(\vec{k})$. It follows from (2), (14), and (15) that the covariance of $f(P)$ is

$$\begin{aligned}\langle f(P)f^*(P')\rangle &\sim \frac{f_o^2 V^2}{4\pi^2 RR'} \, k^4 \, e^{ik\Delta R} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\vec{K}-\vec{k})G(\vec{K}'-\vec{k}') \langle dN(\vec{k})dN^*(\vec{k}')\rangle \\ &= \frac{f_o^2 V^2 \langle(\Delta n)^2\rangle}{4\pi^2 RR'} \, k^4 \, e^{ik\Delta R} \int_{-\infty}^{\infty} G(\vec{K}-\vec{k})G(\vec{K}'-\vec{k}) \, \underline{\Phi}(\vec{k})d\vec{k} \quad .\end{aligned} \tag{16}$$

Again, we require that (16) be a good approximation only when the magnitude of $\langle f(P)f^*(P')\rangle$ is an appreciable fraction of its value for $P = P'$, i.e., only when the functions $G(\vec{K}-\vec{k})$ and $G(\vec{K}'-\vec{k})$ overlap appreciably. The overlap condition is roughly that $|\vec{K}-\vec{K}'| \lesssim (2\pi/d)$. Since it is easily established that the form factor obeys the composition rule

$$\int_{-\infty}^{\infty} G(\vec{K}-\vec{k})G(\vec{K}'-\vec{k}) \, d\vec{k} = \frac{(2\pi)^3}{V} \, G(\vec{K}-\vec{K}') \quad ,$$

it follows that if $\underline{\Phi}(\vec{k})$ is substantially constant in the range $|\vec{K}-\vec{K}'| \lesssim (2\pi/d)$, which means that (11) is valid, then we can write

$$(17) \qquad \int_{-\infty}^{\infty} G(\vec{K}-\vec{k})G(\vec{K}'-\vec{k})\,\underline{\Phi}(\vec{k})d\vec{k} \sim \frac{(2\pi)^3}{V}\,\underline{\Phi}(\vec{K})G(\vec{K}-\vec{K}') \sim \frac{(2\pi)^3}{V}\,\underline{\Phi}(\vec{K}')G(\vec{K}-\vec{K}')$$

$$\sim \frac{(2\pi)^3}{V}\,\underline{\Phi}\left(\frac{\vec{K}+\vec{K}'}{2}\right) G(\vec{K}-\vec{K}') \quad .$$

The derivation of (10), (10a), and (12) is completed by substituting (17) in (16).

3. Scattering by locally homogeneous dielectric noise

As an alternative approach to the representation of a confined region of dielectric noise as a three-dimensional sample of a homogeneous process, we can incorporate the feature of being spatially confined into the process $\Delta n(\vec{r})$ itself. Thus, guided by the one-dimensional theory of locally stationary random processes [1], we say that the dielectric noise is (second order) locally homogeneous if the covariance of $\Delta n(\vec{r})$ is of the form

$$(18) \qquad \langle \Delta n(\vec{r})\,\Delta n(\vec{r}') \rangle = \left\langle \left(\Delta n\left(\frac{\vec{r}+\vec{r}'}{2}\right)\right)^2 \right\rangle C(\vec{r}-\vec{r}') \quad ,$$

where $\left\langle \left(\Delta n\left(\frac{\vec{r}+\vec{r}'}{2}\right)\right)^2 \right\rangle$ is a non-negative function, and $C(\vec{r}-\vec{r}')$ is a correlation function (i.e., a homogeneous covariance). Covariances of the form (18) will be called locally homogeneous covariances. For convenience, we introduce the new variables

$$(19) \qquad \vec{\rho}_+ = \frac{\vec{r}+\vec{r}'}{2} \quad ,$$

$$\vec{\rho}_- = \vec{r}-\vec{r}' \quad .$$

The quantity $\langle (\Delta n(\vec{\rho}_+))^2 \rangle$ has the meaning of a sliding power factor which renormalizes the average instantaneous power of the process $\Delta n(\vec{r})$ to a representative local power level, chosen as the power level at the centroid of the points

$\vec{r}$ and $\vec{r}'$. We assume that

$$\int_{-\infty}^{\infty} \langle (\Delta n(\vec{\rho}_+))^2 \rangle \, d\vec{\rho}_+ < \infty \quad , \tag{20}$$

i.e., the integrated average power of the process $\Delta n(\vec{r})$ is finite. When $\langle (\Delta n(\vec{\rho}_+))^2 \rangle$ is taken to be the positive constant $\langle (\Delta n)^2 \rangle$, equation (18) reduces to the expression (1) for the case of homogeneous dielectric noise.* In physical situations where the average noise power is a function of position, but is slowly varying with respect to the correlation distance of the process, it should be an excellent approximation to write the covariance of the process in the form (18). This is typical of the state of affairs when the dielectric noise is produced by the action of turbulence. In fact, our definition of local homogeneity stems from a need to give mathematical form to Kolmogoroff's qualitative definition of local homogeneity [6], which expresses the physical fact that the turbulent regime can vary from one large eddy to the next.

The question of the existence of covariances of the form (18) is non-trival, and is examined in [1]. For our purposes, it will suffice to give two examples.

a. The <u>binormal covariance</u>

$$\exp\left[-4(\rho_+/d)^2\right] \quad \exp\left[-(\rho_-/d)^2\right] \tag{21}$$

is locally homogeneous. This follows at once from the fact that

$$e^{-2(r/d)^2} \, e^{-2(r'/d)^2} = e^{-4(\rho_+/d)^2} \, e^{-(\rho_-/d)^2} \quad ,$$

and the fact that $\exp(-x^2)$ is both non-negative and a correlation function.

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*In the case of homogeneous processes (20) is not valid, corresponding to the fact that homogeneous processes have the physically unreasonable property of not being spatially confined.

Incidentally, this example shows that for a process to be locally homogeneous, it is not necessary for the average noise power to be slowly varying with respect to the correlation distance, since both factors in (21) have approximately the same width.

b. It follows from the closure of the class of covariances under multiplication that the product of the binormal covariance (20) and any correlation function is locally homogeneous. In particular, if we multiply the binormal covariance by any Gaussian correlation function $\exp\left[-(\rho_-/d_1)^2\right]$, we obtain a locally homogeneous covariance. Writing

$$e^{-(\rho_-/d)^2}\ e^{-(\rho_-/d_1)^2} = e^{-(\rho_-/d')^2} ,$$

we have

$$(1/d')^2 = (1/d)^2 + (1/d_1)^2 ,$$

so that

$$d'^2 \le d^2 , \qquad d'^2 \le d_1^2 .$$

It follows that the more general form of the binormal covariance

$$e^{-4(\rho_+/d)^2}\ e^{-(\rho_-/d')^2} , \qquad d' \le d , \tag{22}$$

is also locally homogeneous.

Consider now the scattering of plane waves by locally homogeneous dielectric noise (18), and suppose that most of the contribution to the integral $\int_{-\infty}^{\infty} \langle(\Delta n(\vec{\rho}_+))^2\rangle\, d\vec{\rho}_+$ comes from a certain finite region V. (For example, for the binormal covariance (22), V is a sphere of radius $\sim d$.) Then it is a good approximation to write the scattered field observed at a point P in the far field of V as

$$f(P) = -\frac{f_o}{2\pi R}\, k^2\, e^{ikR} \int_{-\infty}^{\infty} e^{i\vec{K}\cdot\vec{r}}\, \Delta n(\vec{r})\, d\vec{r} . \tag{23}$$

The only difference between (23) and (2) is that the integral extends over all

space instead of over a finite scattering volume V. Correspondingly, the covariance of f(P) is given by

$$\langle f(P)f^*(P')\rangle = \frac{f_o^2}{4\pi^2 RR'} k^4 e^{ik\Delta R} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\vec{K}\cdot\vec{r}-\vec{K}'\cdot\vec{r}')}\langle \Delta n(\vec{r})\,\Delta n(\vec{r}')\rangle \, d\vec{r}d\vec{r}',$$

which is to be compared with (4). However, because of the infinite limits, it is now a simple matter to evaluate the double integral exactly. Substituting from (18), we find that the double integral becomes

$$(24)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[i\left(\frac{\vec{K}+\vec{K}'}{2}\right)\cdot\vec{\rho}_-\right]\exp\left[i(\vec{K}-\vec{K}')\cdot\vec{\rho}_+\right]\langle(\Delta n(\vec{\rho}_+))^2\rangle C(\vec{\rho}_-)\, d\vec{\rho}_+\, d\vec{\rho}_-$$

$$= \int_{-\infty}^{\infty} \exp\left[i\left(\frac{\vec{K}+\vec{K}'}{2}\right)\cdot\vec{\rho}_-\right] C(\vec{\rho}_-)d\vec{\rho}_- \int_{-\infty}^{\infty} \exp\left[i(\vec{K}-\vec{K}')\cdot\vec{\rho}_+\right]\langle(\Delta n(\vec{\rho}_+))^2\rangle d\vec{\rho}_+ ,$$

where the new variables $\vec{\rho}_+$ and $\vec{\rho}_-$ are defined by (19).* By (8a), the first integral in the right side of (24) is just $(2\pi)^3 \Phi\left(\frac{\vec{K}+\vec{K}'}{2}\right)$. We now define a new form factor

$$H(\vec{k}) = \left[\int_{-\infty}^{\infty}\langle(\Delta n(\vec{\rho}_+))^2\rangle d\vec{\rho}_+\right]^{-1}\int_{-\infty}^{\infty} e^{i\vec{k}\cdot\vec{\rho}_+}\langle(\Delta n(\vec{\rho}_+))^2\rangle d\vec{\rho}_+ ,$$

which is the Fourier transform of a non-negative function, and hence a correlation function. The covariance of f(P) can then be written as

$$\langle f(P)f^*(P')\rangle = \frac{2\pi f_o^2 \int_{-\infty}^{\infty}\langle(\Delta n(\vec{\rho}_+))^2\rangle d\vec{\rho}_+}{RR'} k^4 e^{ik\Delta R}\Phi\left(\frac{\vec{K}+\vec{K}'}{2}\right) H(\vec{K}-\vec{K}') ,$$

which is to be compared with (12). We note that the quantity $\int_{-\infty}^{\infty}\langle(\Delta n(\vec{\rho}_+))^2\rangle d\vec{\rho}_+$ plays the role of the quantity $V\langle(\Delta n)^2\rangle$ in (12), as is to be expected. The form factor $H(\vec{K}-\vec{K}')$, which describes the space correlations of the received radiation, may be regarded as the Fraunhofer diffraction pattern associated with the non-uniform dielectric noise power $\langle(\Delta n(\vec{\rho}_+))^2\rangle$. The fact that a finite sample of a homogeneous process can be considered an approximation to a locally homogeneous process is now seen to be the motivation for our choice of the symmetric

*The quantity (24) is itself a locally homogeneous covariance; see [1].

combination $(\vec{K}+\vec{K}')/2$ in the approximation (12). Finally, we note that the average scattered power $\langle |f(P)|^2 \rangle$ received at a fixed point P is proportional to $\underline{\Phi}(\vec{K})$, as in Section 2.

4. Conclusion

We have seen how to express rigorously processes which, although not homogeneous in the large, are locally homogeneous. When plane waves are scattered by locally homogeneous dielectric noise and observed in the Fraunhofer region, it is found that the local structure of the noise determines the average scattered power received at a fixed point, whereas the overall structure of the noise determines the space correlations of the radiation received at two different points. It would be worthwhile to try to develop a theory for the case where the observation points are not in the Fraunhofer region, since, in typical radio scattering experiments, the antenna beams are not narrow enough to meet the Fraunhofer condition [4].

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AUG 2 2	LAM	

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